HARRISBURG
CHRISTIAN SCHOOL

Summer Math

for students entering

Algebra III / Trigonometry

REVIEW LESSON A *Fractional equations · Radical equations · Linear systems · Exponents and radicals*

RLA.A Fractional equations

A good first step in the solution of a fractional equation is to multiply the numerator in each term on both sides of the equation by the least common multiple of the denominators. This will permit the denominators to be eliminated, and the resulting equation can then be solved. Of course, values of the variable that would cause one or more denominators to equal zero are unacceptable as solutions. In the following example, x cannot equal -2 because replacing x with -2 would cause the second denominator to equal zero.

Example RLA.A.1 Solve: $\frac{4}{7} + \frac{3}{x+2} = \frac{5}{3}$

Solution ($x \neq -2$). We have noted the unacceptable value of the variable. Next we multiply each numerator by $21(x + 2)$, which is the least common multiple of the denominators. Then we cancel the denominators and solve.

$$21(x+2) \cdot \frac{4}{7} + 21(x+2) \cdot \frac{3}{x+2} = 21(x+2) \cdot \frac{5}{3} \quad \text{multiplied}$$

$$3(x+2)4 + 21 \cdot 3 = 7(x+2)5 \quad \text{canceled}$$

$$12x + 24 + 63 = 35x + 70 \quad \text{multiplied}$$

$$23x = 17 \quad \text{simplified}$$

$$x = \frac{17}{23} \quad \text{divided}$$

Example RLA.A.2 Solve:

$$\begin{cases} (a) & \frac{3}{7}x + \frac{2}{5}y = 11 \\ (b) & 0.03x - 0.2y = -0.37 \end{cases}$$

Solution If we multiply the top equation by 35, we can eliminate the denominators. If we multiply the bottom equation by 100, we can make all the numbers in the bottom equation whole numbers.

	MULTIPLYING		SIMPLIFIED EQUATIONS	
(*a*)	$35 \cdot \frac{3}{7}x + 35 \cdot \frac{2}{5}y = 35 \cdot 11$	$\longrightarrow$	$15x + 14y = 385$	(*c*)
(*b*)	$100(0.03x) - 100(0.2y) = 100(-0.37)$	$\longrightarrow$	$3x - 20y = -37$	(*d*)

Now if we multiply equation (d) by -5, we can add the equations and eliminate x.

(c) $15x + 14y = 385$	$\longrightarrow$ (1)	$\longrightarrow$	$15x + 14y = 385$	multiplied by 1
(d) $3x - 20y = -37$	$\longrightarrow$ (-5)	$\longrightarrow$	$-15x + 100y = 185$	multiplied by -5
			$114y = 570$	added
			$y = \dfrac{570}{114}$	transformed and divided
			$\mathbf{y = 5}$	solved

Now we use equation (c) to solve for x.

$15x + 14y = 385$	equation (c)
$15x + 14(5) = 385$	substituted 5 for y
$15x + 70 = 385$	multiplied
$x = \dfrac{315}{15}$	transformed and divided
$\mathbf{x = 21}$	solved

So our solution is the ordered pair **(21, 5)**.

RLA.B Radical equations

Radicals in equations can be eliminated by isolating the radical on one side of the equation and then raising both sides of the equation to the power that will eliminate the radical. When the radical is a square root radical, both sides are raised to the second power. When the radical is a cube root radical, both sides are raised to the third power, and so on. **When an equation contains two radicals, it is sometimes necessary to repeat the procedure.**

Example RLA.B.1 Solve: $\sqrt[3]{x-5} - 2 = 2$

Solution We first isolate the radical by adding $+2$ to both sides of the equation. To eliminate the cube root radical, we raise both sides of the equation to the third power. Then we solve the resulting equation.

$\sqrt[3]{x-5} = 4$	added $+2$ to both sides
$x - 5 = 64$	raised both sides to third power
$\mathbf{x = 69}$	solved

Now we check our answer in the original equation because raising expressions to a power sometimes generates an equation that has spurious solutions.

$\sqrt[3]{(69) - 5} = 4$	substituted
$\sqrt[3]{64} = 4$	simplified
$4 = 4$	check

Example RLA.B.2 Solve: $\sqrt{s-48} + \sqrt{s} = 8$

Solution First we isolate $\sqrt{s-48}$ and square both sides to eliminate this radical.

$\sqrt{s-48} = 8 - \sqrt{s}$	added $-\sqrt{s}$ to both sides
$s - 48 = 64 - 16\sqrt{s} + s$	squared both sides
$-48 = 64 - 16\sqrt{s}$	simplified

Now we rearrange the equation to isolate $\sqrt{s}$, and then we square both sides of the equation again.

$$-112 = -16\sqrt{s} \quad \text{rearranged}$$
$$7 = \sqrt{s} \quad \text{divided both sides by } -16$$
$$49 = s \quad \text{squared both sides}$$

We finish by checking our solution in the original equation.

$$\sqrt{(49) - 48} + \sqrt{(49)} = 8 \quad \text{substituted}$$
$$\sqrt{1} + 7 = 8 \quad \text{simplified}$$
$$8 = 8 \quad \text{check}$$

Example RLA.B.3 Solve: $x^{2/3} = 4$

Solution **This problem gives us another example of how we get rid of undesirable exponents.** We want to find the value of x^1 so we raise both sides of the equation to the 3/2 power.

$$(x^{2/3})^{3/2} = 4^{3/2} \quad \text{raised to 3/2 power}$$
$$x = 8 \quad \text{simplified}$$

RLA.C Systems of three linear equations

Systems of linear equations can be solved by using either the substitution method or the elimination method. Some systems are most easily solved if both methods are used.

Example RLA.C.1 Solve:

$$\begin{cases} (a)\ 2x + 2y - z = 14 \\ (b)\ 3x + 3y + z = 16 \\ (c)\ x - 2y = 0 \end{cases}$$

Solution In this book, systems of equations of three or more unknowns will be designed so that the numbers are easy to handle and so that most of the answers will be integers. These problems are studied to allow practice in the concepts of solving equations, and not for practice in arithmetic. Even though the numbers in these problems are integers, a calculator can often be used to prevent mistakes in arithmetic. For the first step, we solve equation (c) for x and find that $x = 2y$. Then in equations (a) and (b), we replace x with $2y$, simplify, and use elimination to solve for y.

$$(a)\ 2(2y) + 2y - z = 14 \longrightarrow 6y - z = 14 \quad \text{substituted } 2y \text{ for } x$$
$$(b)\ 3(2y) + 3y + z = 16 \longrightarrow 9y + z = 16 \quad \text{substituted } 2y \text{ for } x$$
$$15y = 30 \quad \text{added}$$
$$y = 2 \quad \text{solved}$$

Now we know that $x = 4$ because equation (c) tells us that $x = 2y$. We finish by using 2 for y and 4 for x in equation (a), and solving for z.

$$2(4) + 2(2) - z = 14 \quad \text{substituted 2 for } y \text{ and 4 for } x$$
$$12 - z = 14 \quad \text{simplified}$$
$$z = -2 \quad \text{solved}$$

Thus, the solution is the ordered triple **(4, 2, −2)**.

Example RLA.C.2 Solve:

$$\begin{cases} x + 2y + z = 4 & (a) \\ 2x - y - z = 0 & (b) \\ 2x - 2y + z = 1 & (c) \end{cases}$$

Solution We decide to begin by eliminating z. Thus, we first add equations (a) and (b) to get equation (d), which has no z term. Now we must use equation (c). We can add equation (c) to either equation (a) or (b) and eliminate z again. We decide to add equation (c) to equation (b).

$$\begin{array}{lrcl} (a) & x + 2y + z &=& 4 \\ (b) & 2x - y - z &=& 0 \\ \hline (d) & 3x + y &=& 4 \end{array} \qquad \begin{array}{lrcl} (b) & 2x - y - z &=& 0 \\ (c) & 2x - 2y + z &=& 1 \\ \hline (e) & 4x - 3y &=& 1 \end{array}$$

Now we have the two equations (d) and (e) in the two unknowns x and y. We will use elimination and add equation (e) to the product of equation (d) and 3.

$$\begin{array}{lrcl} (3)(d) & 9x + 3y &=& 12 \\ (e) & 4x - 3y &=& 1 \\ \hline & 13x &=& 13 \\ & & & \mathbf{x = 1} \end{array}$$

Now we can use $x = 1$ in either equation (d) or (e) to find y. This time we will do both to show that either procedure will yield the same result.

$$\begin{array}{lrcl} (d) & 3(1) + y &=& 4 \\ & 3 + y &=& 4 \\ & y &=& \mathbf{1} \end{array} \qquad \begin{array}{lrcl} (e) & 4(1) - 3y &=& 1 \\ & 4 - 3y &=& 1 \\ & -3y &=& -3 \\ & y &=& \mathbf{1} \end{array}$$

Now we can use $x = 1$ and $y = 1$ in either (a), (b), or (c) to find z. This time we will use all three equations to show that all three will produce the same result.

USING EQUATION (a)	USING EQUATION (b)	USING EQUATION (c)
$(1) + 2(1) + z = 4$	$2(1) - (1) - z = 0$	$2(1) - 2(1) + z = 1$
$1 + 2 + z = 4$	$2 - 1 - z = 0$	$2 - 2 + z = 1$
$3 + z = 4$	$1 - z = 0$	$0 + z = 1$
$\mathbf{z = 1}$	$\mathbf{z = 1}$	$\mathbf{z = 1}$

Thus, the solution is the ordered triple **(1, 1, 1)**.

Example RLA.C.3 Solve:

$$\begin{cases} 2x + 3y = -4 & (a) \\ x - 2z = -3 & (b) \\ 2y - z = -6 & (c) \end{cases}$$

Solution One variable is missing in each equation. We can see this better if we write the equations in expanded form.

$$\begin{array}{lrrrcl} (a) & 2x & + 3y & &=& -4 \\ (b) & x & & - 2z &=& -3 \\ (c) & & 2y & - z &=& -6 \end{array}$$

The first step is to combine any two of the equations so that either x, y, or z is eliminated. We could use equations (a) and (b) and eliminate x; or use equations (b) and (c) and eliminate z; or use equations (a) and (c) and eliminate y. We choose to eliminate x, so we add equation (a) to the product of equation (b) and -2.

$$\begin{array}{rrrrr} (a) & 2x + 3y & & = & -4 \\ (-2)(b) & -2x & + 4z & = & 6 \\ \hline (d) & 3y & + 4z & = & 2 \end{array}$$

The resulting equation, (d), has y and z as variables. So does equation (c). We use these equations to eliminate z by adding equation (d) to the product of (4) and equation (c).

$$\begin{array}{rrr} (4)(c) & 8y - 4z = & -24 \\ (d) & 3y + 4z = & 2 \\ \hline & 11y \quad = & -22 \\ & y = & -2 \end{array}$$

Now we replace y with -2 in equation (a) and solve for x. Then we replace y with -2 in equation (c) and solve for z.

$$\begin{array}{rr} (a) & 2x + 3(-2) = -4 \\ & 2x - 6 = -4 \\ & 2x = 2 \\ & x = 1 \end{array} \qquad \begin{array}{rr} (c) & 2(-2) - z = -6 \\ & -4 - z = -6 \\ & -z = -2 \\ & z = 2 \end{array}$$

Thus, our solution is the ordered triple $(\mathbf{1}, \mathbf{-2}, \mathbf{2})$.

RLA.D Exponents and radicals

When we simplify radical expressions, it is often helpful to replace radicals with parentheses and fractional exponents.

Example RLA.D.1 Simplify: $\sqrt{x^3y}\,\sqrt[4]{xy^3}$

Solution First we replace the radicals with parentheses and fractional exponents and multiply exponents where indicated.

$$(x^3y)^{1/2}(xy^3)^{1/4} = x^{3/2}y^{1/2}x^{1/4}y^{3/4}$$

Now we rearrange the bases and simplify by adding the exponents of like bases.

$$x^{3/2}x^{1/4}y^{1/2}y^{3/4} = x^{7/4}y^{5/4}$$

Example RLA.D.2 Simplify: $\dfrac{a^{x/2}(y^{2-x})^{1/2}}{a^{3x}y^{-2x}}$

Solution We simplify and write all exponentials in the numerator.

$$a^{x/2}a^{-3x}y^{1-x/2}y^{2x}$$

We finish by adding the exponents of like bases and get

$$a^{-5x/2}y^{(1+3x)/2}$$

Example RLA.D.3 Simplify: $3\sqrt{\dfrac{3}{2}} - 4\sqrt{\dfrac{2}{3}} + 2\sqrt{24}$

Solution First we change the form of the radicals and use multiplication as necessary to rationalize the denominators.

$$3\frac{\sqrt{3}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} - 4\frac{\sqrt{2}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} + 2\sqrt{4}\sqrt{6}$$

Next we simplify.

$$\frac{3\sqrt{6}}{2} - \frac{4\sqrt{6}}{3} + 4\sqrt{6}$$

We finish by finding a common denominator and adding.

$$\frac{9\sqrt{6}}{6} - \frac{8\sqrt{6}}{6} + \frac{24\sqrt{6}}{6} = \frac{25\sqrt{6}}{6}$$

Example RLA.D.4 Simplify: $(\sqrt{2} + \sqrt{x})(1 - \sqrt{x})$

Solution We multiply as indicated.

$$\begin{array}{r} \sqrt{2} + \sqrt{x} \\ 1 - \sqrt{x} \\ \hline \sqrt{2} + \sqrt{x} \\ - \sqrt{2x} - x \\ \hline \mathbf{\sqrt{2} + (1 - \sqrt{2})\sqrt{x} - x} \end{array}$$

There are three terms in the answer. We note that the coefficient of the second term is the sum $1 - \sqrt{2}$.

Example RLA.D.5 Simplify: $\dfrac{x^{-2} + y^{-2}}{(xy)^{-1}}$

Solution We begin by eliminating the negative exponents and then we add the two expressions in the numerator.

$$\frac{\dfrac{1}{x^2} + \dfrac{1}{y^2}}{\dfrac{1}{xy}} = \frac{\dfrac{y^2 + x^2}{x^2y^2}}{\dfrac{1}{xy}} \qquad \text{added}$$

We finish by multiplying above and below by the reciprocal of the denominator.

$$\frac{\dfrac{y^2 + x^2}{x^2y^2} \cdot \dfrac{xy}{1}}{\dfrac{1}{xy} \cdot \dfrac{xy}{1}} = \mathbf{\frac{x^2 + y^2}{xy}}$$

Problem set A Problems that compare the values of quantities come in many forms and can be used to provide practice in mathematical reasoning. In these problems, a statement will be made about two quantities *A* and *B*. The correct answer is *A* if quantity *A* is greater and is *B* if quantity *B* is greater. The correct answer is *C* if the quantities are equal and is *D* if insufficient information is provided to determine which quantity is greater.

Compare:

1. *A.* $\dfrac{6 + \dfrac{3}{4}}{2 - \dfrac{5}{4}}$ *B.* 3^2

2. $y \neq 0$: *A.* $\dfrac{x + y}{y}$ *B.* $\dfrac{x}{y} + 1$

3. *A.* $\sqrt{\dfrac{1}{4}} + \sqrt{\dfrac{1}{25}}$ *B.* $\sqrt{\dfrac{1}{4} + \dfrac{1}{25}}$

4. If $x < 0$ and $y < 0$, compare: *A.* $x + y$ *B.* $x - y$

5. Twenty percent of the molybdenum fused. If 1420 grams did not fuse, what was the total weight of molybdenum used?

6. The ratio of pusillanimous brave men to oxymorons on the battlefield was 17 to 2. If the total of both on the battlefield was 342, how many were oxymorons?

Solve:

7. $\frac{4}{7} + \frac{3}{x+3} = \frac{5}{3}$

8. $\frac{5}{3} - \frac{2}{x-4} = \frac{1}{2}$

9. $\frac{1}{x-7} + \frac{1}{4} = \frac{1}{3}$

10. $\begin{cases} \frac{3}{7}x + \frac{2}{5}y = 11 \\ 0.03x - 0.2y = -0.37 \end{cases}$

11. $\begin{cases} \frac{2}{3}x + \frac{3}{5}y = 12 \\ 0.1x + 0.02y = 1.1 \end{cases}$

12. $\sqrt[3]{x-5} - 1 = 2$

13. $\sqrt{s-27} + \sqrt{s} = 9$

14. $\sqrt{s-7} + \sqrt{s} = 7$

15. $\begin{cases} 2x + 2y - z = 9 \\ 3x + 3y + z = 16 \\ x - 2y = -1 \end{cases}$

16. $\begin{cases} x - 2y + z = -2 \\ 2x - 2y - z = -3 \\ x + y - 2z = 1 \end{cases}$

17. $\begin{cases} 2x - y + z = 0 \\ 4x + 2y + z = 2 \\ 2x - y - z = -4 \end{cases}$

18. $\begin{cases} 2x + 3y = -1 \\ x - 2z = -3 \\ 2y - z = -4 \end{cases}$

19. $\begin{cases} \frac{x}{2} + \frac{y}{4} = 2 \\ x - \frac{z}{3} = 1 \\ \frac{y}{2} + z = 5 \end{cases}$

20. $\begin{cases} x - y = 1 \\ y - 2z = 1 \\ 3x - 4z = 7 \end{cases}$

Simplify:

21. $\sqrt{x^3y^2}\,\sqrt[4]{xy^3}$

22. $x^{3/4}\sqrt{xy}\,x^{1/2}\sqrt[3]{x^4}$

23. $\frac{a^{x/2}(y^{2-x})^{1/2}}{a^{4x}y^{-2x}}$

24. $\frac{y^{x+3}y^{x/2-1}z^{a}}{y^{(x-a)/2}z^{(x-a)/3}}$

25. $2\sqrt{\frac{3}{2}} - 3\sqrt{\frac{2}{3}} + 2\sqrt{24}$

26. $2\sqrt{\frac{7}{3}} - \sqrt{\frac{3}{7}} - 2\sqrt{84}$

27. $(\sqrt{2} - \sqrt{x})(1 - \sqrt{x})$

28. $(\sqrt{3} + \sqrt{x})(\sqrt{3} + \sqrt{x})$

29. $\sqrt{2x}(\sqrt{3x} + \sqrt{x})$

30. $\frac{x^{-2} + y^{-2}}{(xy)^{-1}}$

REVIEW LESSON B Complex numbers · Rational denominators · Completing the square · The quadratic formula

RLB.A Complex numbers

In mathematics, we use the letter i to represent the positive square root of -1.

$$i = \sqrt{-1}$$

The square root of -1 is encountered often in the solution of quadratic equations. It may be that i was first used to represent $\sqrt{-1}$ because i can be written with one stroke of the pen (if the dot is neglected) while $\sqrt{-1}$ requires three strokes of the pen. Square roots of negative numbers can be written as the product of a real number and i as we show here.

$$\sqrt{-13} = \sqrt{13(-1)} = \sqrt{13}\sqrt{-1} = \sqrt{13}\,i$$

Since $i = \sqrt{-1}$, then $i^2 = -1$.

$$i^2 = -1 \qquad \text{because} \qquad \sqrt{-1}\sqrt{-1} = -1$$

A complex number is made up of two real numbers a and b and the letter i. If the number with i as a factor is written last as $a + bi$, the complex number is said to be in ***standard form.*** A complex number has a real component a and an imaginary component. In the past, the combination bi was considered to be the imaginary component, but recently more than a few authors have said that the real number b is the imaginary component of the complex number. Note that all of the following numbers are complex numbers in the standard form $a + bi$.

(*a*) $4 + 2i$ (*b*) $3.01 - \dfrac{\sqrt{2}}{3}i$ (*c*) 4.06 (*d*) $3i$ (*e*) $\dfrac{\sqrt{13}}{5.6} + \sqrt{2}\,i$

We note that a and b can be any real numbers, including 0, because (*c*) represents $4.06 + 0i$ and (*d*) represents $0 + 3i$.

Example RLB.A.1 Simplify: $3i^3 + 2i^5 - 3i + 2i^2$

Solution First we expand the i terms and pair the i's.

$$3i(ii) + 2i(ii)(ii) - 3i + 2(ii)$$

Now we replace each pair of i's with -1.

$$3i(-1) + 2i(-1)(-1) - 3i + 2(-1)$$

We finish by simplifying and writing the result in standard form.

$$-3i + 2i - 3i - 2 = \mathbf{-2 - 4i}$$

Example RLB.A.2 Simplify: $\sqrt{-3}\sqrt{4} + 3\sqrt{-2}\sqrt{-9} + \sqrt{-16} + \sqrt{16}$

Solution We begin by using the i notation.

$$(\sqrt{3}\,i)(2) + (3\sqrt{2}\,i)(3i) + 4i + 4$$

Now we simplify

$$2\sqrt{3}\,i - 9\sqrt{2} + 4i + 4$$

and finish by grouping the real parts and the imaginary parts.

$$(4 - 9\sqrt{2}) + (2\sqrt{3} + 4)i$$

RLB.B Rational denominators

It is customary to write numbers with rational denominators. For example, the four numbers

$$(a)\ \frac{2}{\sqrt{3}} \qquad (b)\ \frac{6}{2i} \qquad (c)\ \frac{4+\sqrt{3}}{2-3\sqrt{3}} \qquad (d)\ \frac{2-i^3+2i^5}{-2i+4}$$

can be written with rational denominators as

$$(a')\ \frac{2\sqrt{3}}{3} \qquad (b')\ -3i \qquad (c')\ \frac{-17-14\sqrt{3}}{23} \qquad (d')\ \frac{1}{10}+\frac{4}{5}i$$

The process of converting a denominator to a rational number is called ***rationalizing the denominator.*** In (*a*) we multiplied above and below by $\sqrt{3}$. In (*b*) we multiplied above and below by $-i$. We will show the procedure for (*c*) and (*d*) in the next two examples.

Example RLB.B.1 Simplify: $\dfrac{4+\sqrt{3}}{2-3\sqrt{3}}$

Solution We remember that **an expression that contains square roots of counting numbers is in simplified form when no radicand has a perfect square (other than 1) as a factor and no radicals are in the denominator.** We can rationalize the denominator if we multiply above and below by $2 + 3\sqrt{3}$, which is the ***conjugate of the denominator.***

$$\frac{4+\sqrt{3}}{2-3\sqrt{3}} \cdot \frac{2+3\sqrt{3}}{2+3\sqrt{3}}$$

We have two multiplications to perform, one above and one below. Many people find it easier to do these multiplications separately and then to write the answer. We will do this.

ABOVE	BELOW
$4 + \sqrt{3}$	$2 - 3\sqrt{3}$
$2 + 3\sqrt{3}$	$2 + 3\sqrt{3}$
$8 + 2\sqrt{3}$	$4 - 6\sqrt{3}$
$12\sqrt{3} + 9$	$+ 6\sqrt{3} - 27$
$17 + 14\sqrt{3}$	$4 - 27 = -23$

Thus, our simplification is

$$\frac{17+14\sqrt{3}}{-23} \quad \text{or} \quad \frac{-17-14\sqrt{3}}{23}$$

Example RLB.B.2 Simplify: $\dfrac{2-i^3+2i^5}{-2i+4}$

Solution First we write both complex numbers in standard form and get

$$\frac{2+3i}{4-2i}$$

We can change the denominator of this expression to a rational number if we multiply above and below by $4 + 2i$, which is the **conjugate of the denominator.** We have two multiplications indicated, one above and one below. We will use the vertical format for both of the multiplications.

$$\begin{array}{lll} \text{ABOVE} & & \text{BELOW} \\ 2 + 3i & & 4 - 2i \\ 4 + 2i & & 4 + 2i \\ \hline 8 + 12i & & 16 - 8i \\ \quad\; 4i + 6i^2 & & \quad + 8i - 4i^2 \\ \hline 8 + 16i - 6 = 2 + 16i & & 16 \quad\; + 4 = 20 \end{array}$$

Thus, we can write our answer as

$$\frac{2 + 16i}{20} = \frac{1 + 8i}{10}$$

This answer is not in the preferred form of $a + bi$. We can write this complex number in standard form if we write

$$\frac{1}{10} + \frac{4}{5}i$$

RLB.C Completing the square

If a quadratic equation is in the form of the two equations shown here, the solution can be found by taking the square root of both sides of the equation.

(a)	(b)	
$\left(x + \frac{1}{2}\right)^2 = 3$	$(x - 4)^2 = 5$	equation
$x + \frac{1}{2} = \pm\sqrt{3}$	$x - 4 = \pm\sqrt{5}$	square root of both sides
$x = -\frac{1}{2} \pm \sqrt{3}$	$x = 4 \pm \sqrt{5}$	solved for x

The process of rearranging a quadratic equation into the form $(x - h)^2 = p$ is called ***completing the square.*** The basis for this procedure comes from observing the patterns of the coefficients when binomials are squared. Here we give four examples.

(c) $(x + 3)^2 = x^2 + 6x + 9$ (d) $(x - 5)^2 = x^2 - 10x + 25$

(e) $\left(x - \frac{1}{2}\right)^2 = x^2 - x + \frac{1}{4}$ (f) $\left(x + \frac{2}{5}\right)^2 = x^2 + \frac{4}{5}x + \frac{4}{25}$

We note that, in each example, the constant term of the trinomial is a positive number and is the square of one-half the coefficient of the x term in the trinomial. Thus, if we have

$$x^2 + \frac{3}{5}x$$

and want to add a constant so that the result is the square of a binomial, we multiply the coefficient of x by $\frac{1}{2}$ and square the result.

$$\left(\frac{3}{5} \cdot \frac{1}{2}\right)^2 = \frac{9}{100}$$

Now if we add 9/100 to the expression under consideration, we can write the result as the square of a binomial.

$$x^2 + \frac{3}{5}x + \frac{9}{100} \quad \xrightarrow{\text{which can be written as}} \quad \left(x + \frac{3}{10}\right)^2$$

We also note that the constant in the binomial is one-half the coefficient of x in the trinomial.

Example RLB.C.1 Solve $-x + 3x^2 = -5$ by completing the square.

Solution As the first step we write the equation in standard form.

$$3x^2 - x + 5 = 0$$

Then we divide every term by 3 so that the coefficient of x^2 will be 1.

$$x^2 - \frac{1}{3}x + \frac{5}{3} = 0$$

Next we write the parentheses and move the constant term to the right side.

$$\left(x^2 - \frac{1}{3}x \qquad \right) = -\frac{5}{3}$$

Now we multiply the coefficient of x by $\frac{1}{2}$ and square this product.

$$\left(-\frac{1}{3} \cdot \frac{1}{2}\right)^2 = \frac{1}{36}$$

Then we add 1/36 to both sides of the equation.

$$\left(x^2 - \frac{1}{3}x + \frac{1}{36}\right) = -\frac{5}{3} + \frac{1}{36}$$

Now we simplify and solve for x.

$$\left(x - \frac{1}{6}\right)^2 = -\frac{59}{36} \qquad \text{simplified}$$

$$x - \frac{1}{6} = \pm\sqrt{-\frac{59}{36}} \qquad \text{square root of both sides}$$

$$x = \frac{1}{6} \pm \frac{\sqrt{59}}{6}i \qquad \text{solved}$$

RLB.D The quadratic formula

By completing the square, any quadratic equation can be written in the form $(x - h)^2 = p$ and then can be solved by taking the square root of both sides. This process is time-consuming, and it is a little faster to use the quadratic formula. This formula can be derived by completing the square on a general quadratic equation. We begin by writing a general quadratic equation that uses the letters a, b, and c as constants.

$$ax^2 + bx + c = 0$$

Next we give x^2 a unity coefficient by dividing every term by a, and we get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Now we move c/a to the right side and use parentheses on the left side.

$$\left(x^2 + \frac{b}{a}x \qquad \right) = \qquad -\frac{c}{a}$$

Note that we placed the $-c/a$ well to the right of the equals sign. Now we multiply b/a by $\frac{1}{2}$ and square the result.

$$\left(\frac{b}{a}\cdot\frac{1}{2}\right)^2 = \frac{b^2}{4a^2}$$

Next add $b^2/4a^2$ inside the parentheses and also to the other side of the equation. On the right we are careful to place $b^2/4a^2$ in front of $-c/a$.

$$\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) = \frac{b^2}{4a^2} - \frac{c}{a}$$

Next we write the parentheses term as a squared term and combine $b^2/4a^2$ and $-c/a$.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Finally, we take the square root of both sides and then solve for x.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad \text{took square roots}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{solved for } x$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{added}$$

The derivation of the quadratic formula will be required in future problem sets. This derivation requires only simple algebraic manipulations, and the requirement that a student be able to perform this derivation is not unreasonable.

Example RLB.D.1 Use the quadratic formula to find the roots of the equation $3x^2 - 2x + 5 = 0$.

Solution The formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If we write the given equation just below the general quadratic equation,

$$ax^2 + bx + c = 0 \qquad \text{general equation}$$

$$3x^2 - 2x + 5 = 0 \qquad \text{given equation}$$

we note the following correspondences between the equations.

$$a = 3 \qquad b = -2 \qquad c = 5$$

If we use these numbers for a, b, and c in the quadratic formula, we get

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(5)}}{2(3)}$$

$$x = \frac{2 \pm \sqrt{-56}}{6}$$

$$x = \frac{1}{3} \pm \frac{\sqrt{14}}{3}i$$

Problem set B

1. Forty percent of those who attended wore red hats. If 1800 of those who attended did not wear red hats, how many did wear red hats?

2. Two animals out of seven believed Chicken Little. If 85 animals did not believe Chicken Little, how many animals were there in all?

Simplify:

3. $3i^4 + 2i^5 + 3i^3 + 2i^2$

4. $\sqrt{-3}\sqrt{4} + 3\sqrt{-2}\sqrt{-9} + \sqrt{-16} + \sqrt{16}$

5. $2i^3 + 3i^2 + 2i - 2\sqrt{2}\,i$

6. $2\sqrt{-2}\sqrt{2} + 3i\sqrt{2} - \sqrt{-2}\sqrt{2}$

7. $\dfrac{2 + i - 2i^3}{2i^3 + 4}$

8. $\dfrac{4 + \sqrt{3}}{2 - 3\sqrt{3}}$

9. $\dfrac{3 - i^2 + 2i^5 - 2}{-2i^3 + 3i}$

Solve by completing the square:

10. $x + 3x^2 = -5$

11. $-2x - 2x^2 = 5$

12. Begin with $ax^2 + bx + c = 0$ and derive the quadratic formula by completing the square.

Use the quadratic formula to solve:

13. $3x^2 + 2x + 5 = 0$

14. $-x = -3x^2 + 4$

15. $-3x = 2x^2 + 7$

16. If $x = 0$, $y > 1$, and $z > 1$, compare: *A.* $2y(x + z)$ *B.* $x(y + z)$

17. If $a = 3$ and $b = \frac{1}{6}$, compare: *A.* $2a - 18b$ *B.* $3a - 36b$

Solve:

18. $\dfrac{2}{x + 4} - \dfrac{1}{5} = \dfrac{2}{3}$

19. $\begin{cases} \frac{2}{3}x - \frac{4}{5}y = 0 \\ 0.02x - 0.1y = -0.76 \end{cases}$

20. $\sqrt{2s - 7} + \sqrt{25} = 7$

21. $\begin{cases} x + 2y - z = 9 \\ x + 3y = 9 \\ 2x - z = 8 \end{cases}$

22. $\begin{cases} x - y + 2z = 7 \\ 2x + y - z = 0 \\ x + 2y + z = 9 \end{cases}$

23. $\begin{cases} \frac{x}{2} - \frac{y}{3} = \frac{5}{6} \\ \frac{2x}{3} + \frac{z}{5} = \frac{11}{5} \\ 0.1y + 0.02z = 0.22 \end{cases}$

24. $\begin{cases} x + y - z = 2 \\ 2x - y + z = 4 \\ 3x + 2y - z = 5 \end{cases}$

Simplify:

25. $x^{2/3}\sqrt[3]{x^2}\,y(x^2y)^{1/5}$

26. $y^{x/2+1}\sqrt{y^a}\,2y^{2a+3}$

27. $2\sqrt{\dfrac{3}{2}} - 4\sqrt{\dfrac{2}{3}} + 3\sqrt{24}$

28. $(\sqrt{2} + \sqrt{x})(\sqrt{2} + \sqrt{x})$

29. $(x^{1/2} + y^{1/2})^2$

30. $\dfrac{a^{-2} + b^{-1}}{a^{-1}b}$

REVIEW LESSON C *Sine, cosine, tangent · Angles of elevation and depression ·*

RLC.A Sine, cosine, and tangent

A right triangle has one right angle and two acute angles. If we select one of the acute angles, we call the ratio of the side opposite this angle to the hypotenuse the ***sine*** of the angle. The ratio of the side adjacent the selected angle to the hypotenuse is called the ***cosine*** of the angle. The ratio of the side opposite the selected angle to the side adjacent the selected angle is called the ***tangent*** of the angle. These ratios are the same for every acute angle whose measure is the same as that of the selected angle. To remember which ratio corresponds to which name requires pure memorization, and mnemonics are helpful for memorizing these ratios. On the right, below, we use the first letters of the words *opposite, hypotenuse,* and *adjacent* to form the first letters of a sentence that is easy to remember.

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \frac{\text{Oscar}}{\text{had}}$$

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \frac{\text{a}}{\text{hold}}$$

$$\tan A = \frac{\text{opposite}}{\text{adjacent}} \qquad \frac{\text{on}}{\text{Arthur}}$$

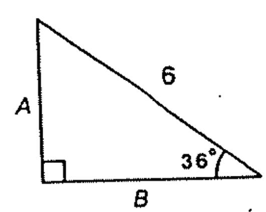
Thus, if we can remember to write sine, cosine, and tangent in that order and then write "Oscar had a hold on Arthur," we have the definitions memorized. Some people take the first letters of the words sine, opposite, hypotenuse; cosine, adjacent, hypotenuse; tangent, opposite, adjacent to form the expression

Soh Cah Toa

and say that it sounds like an American Indian phrase.

We can use the sine, cosine, and tangent ratios to solve for unknown values in right triangles, as we show in the following examples.

Example RLC.A.1 Find A and B.



Solution We can find A and B by using the sine and cosine of 36°.

$$\frac{A}{6} = \sin 36° \qquad \frac{B}{6} = \cos 36°$$

$$\text{so} \quad A = 6 \sin 36° \qquad \text{so} \quad B = 6 \cos 36°$$

Next we use the table of trigonometric functions in the appendix to find the values of sin 36° and cos 36° and make these substitutions to get

$$A = 6(0.5878) \qquad B = 6(0.8090)$$

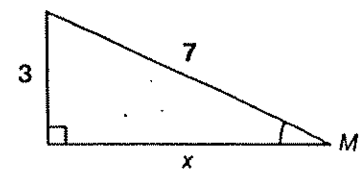
$$A = 3.5268 \qquad B = 4.854$$

A scientific calculator should be used for these problems, and the tables of trigonometric functions should be used when a scientific calculator is not available. Use your calculator to check the following operations.

$$A = 6 \sin 36° \qquad B = 6 \cos 36°$$
$$\approx 3.5267115 \qquad \approx 4.854102$$
$$\approx 3.53 \qquad \approx 4.85$$

When we use the calculator, we can get the answer without having to write the value of sin 36° as an intermediate step. Also, the calculator answer is accurate to more digits, but we will normally round off answers to a more convenient number, as we did here.

Example RLC.A.2 Find angle M and side x.



Solution We use the sine ratio to find angle M.

$$\sin M = \frac{3}{7}$$
$$\sin M = 0.4285714$$
$$M = 25.376932°$$
$$M = \mathbf{25.4°}$$

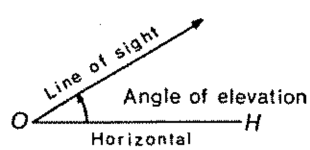
We used the calculator and then rounded off to a more reasonable number. To find side x, we could use a trigonometric function of 25.4° or we could use the theorem of Pythagoras. We decide to use the theorem of Pythagoras.

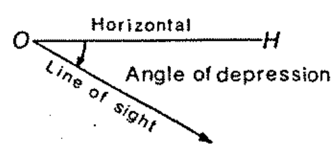
$$x^2 + 3^2 = 7^2$$
$$x^2 = 49 - 9$$
$$x = \sqrt{40} \approx \mathbf{6.32}$$

We used the calculator and rounded off.

RLC.B Angles of elevation and depression

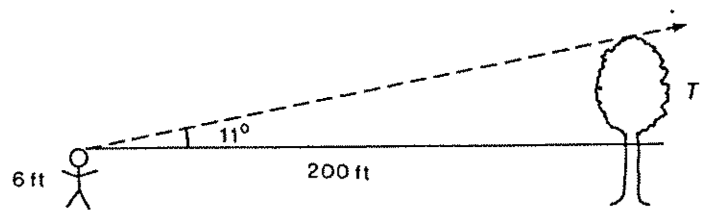
Angles of elevation and **angles of depression** are measured from the horizontal. On the left we show that an angle of elevation is measured ***upward*** from the horizontal, and on the right we show that an angle of depression is measured ***downward*** from the horizontal.





Example RLC.B.1 A man 6 ft tall is 200 ft from a tree, and he measures the angle of elevation to the top of the tree as 11°. How tall is the tree?

Solution **For a problem like this one, a sketch of the problem is very helpful.** This sketch is not drawn to scale.



We solve to find T.

$$\frac{T}{200} = \tan 11°$$

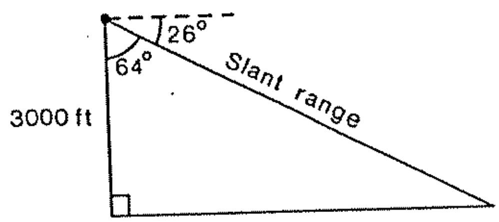
$$T = 200 \tan 11° = 38.88$$

The man was 6 ft tall, so the total height of the tree is

$$\text{Height} = 38.88 \text{ ft} + 6 \text{ ft} = 44.88 \text{ ft}$$

Example RLC.B.2 An airplane is flying at an altitude of 3000 ft above the ground. The pilot sights an object on the ground at an angle of depression of 26°. What is the slant range from the airplane to the object?

Solution The angle of depression is 26°, so the angle in the triangle is 90° − 26° = 64°.



$$\cos 64° = \frac{3000}{s}$$

$$s = \frac{3000}{\cos 64°} = 6844 \text{ ft}$$

We used the calculator to do the arithmetic and rounded off the calculator answer.

Problem set C

1. The ratio of goods to bads was 2 to 5 and the number of bads exceeded twice the number of goods by 40. How many were good and how many were bad?
2. Ten percent of the reds were added to twenty percent of the blues, and the total was 24. Yet, the product of the number of reds and 3 exceeded the number of blues by 20. How many were red and how many were blue?
3. An airplane is flying at an altitude of 5000 ft above the ground. The pilot sights an object on the ground at an angle of depression of 28°. What is the slant range from the airplane to the object?

Simplify:

8. $\sqrt{-3}\sqrt{3} - \sqrt{2}i - \sqrt{-3}\sqrt{-2} - 2$

9. $\dfrac{2 + 2\sqrt{3}}{1 - \sqrt{3}}$

10. $\dfrac{2i^3 - 2i^2 - 1}{i^3 - 2i^4 + 2}$

11. $\dfrac{3i + 2 - 2i^3}{2i - 3i^5 + 2}$

Solve by completing the square:

12. $2x^2 = -x - 5$

13. $2x + 7 = 3x^2$

14. Begin with $ax^2 + bx + c = 0$ and derive the quadratic formula.

Use the quadratic formula to solve:

15. $2x^2 = x - 5$

16. $4 + 3x^2 = -2x$

17. If $0 < x < 10$ and $0 < y < 12$, compare: *A.* x *B.* $y - 2$

18. If $1 < x < 5$ and $1 < y < 5$, compare: *A.* $x - y$ *B.* $y - x$

Solve:

19. $\begin{cases} \frac{4}{5}x - \frac{2}{3}y = \frac{7}{30} \\ 0.01x + 0.1y = 0.03 \end{cases}$

20. $\sqrt{3x - 5} + \sqrt{3x} = 5$

21. $\begin{cases} x + 2y + z = -1 \\ x + z = 3 \\ 3x + y = 4 \end{cases}$

22. $\begin{cases} 2x - y + z = 5 \\ x + y - z = 4 \\ -x + 2y + z = 5 \end{cases}$

23. $\begin{cases} \frac{x}{2} - \frac{y}{3} = \frac{1}{6} \\ \frac{x}{3} + \frac{2z}{5} = 2\frac{3}{5} \\ \frac{y}{3} + \frac{2z}{3} = 4 \end{cases}$

24. $\begin{cases} x + 2y - z = 7 \\ x - 2z = 2 \\ y + 3z = 5 \end{cases}$

Simplify:

25. $a^2\sqrt[5]{a^3b}(a^2b^4)^{1/3}$

26. $\dfrac{x^{a/2-4}y^{(b-3)/2}}{x^{2a}(y^{1/2})^a}$

27. $3\sqrt{\dfrac{7}{2}}-2\sqrt{\dfrac{2}{7}}+3\sqrt{56}$

28. $(x^{1/2}+y^{1/2})(x^{1/2}-y^{-1/2})$

29. $(\sqrt{2}-x)(2-\sqrt{x})$

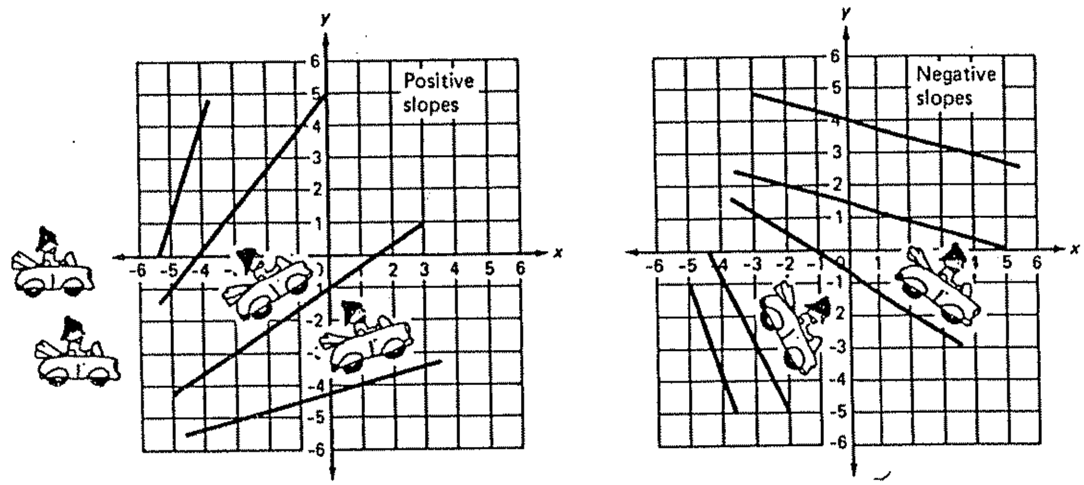
30. $\dfrac{x^{-1}+y^{-1}}{x^{-1}y}$

REVIEW LESSON D *Equation of a line · Complex fractions · Abstract equations · Division of polynomials*

RLD.A Equation of a line

A line that is not vertical has a slope and a y intercept.[†] **If you can determine the slope of a line and know where the line crosses the y axis, the line is defined and can be drawn.**

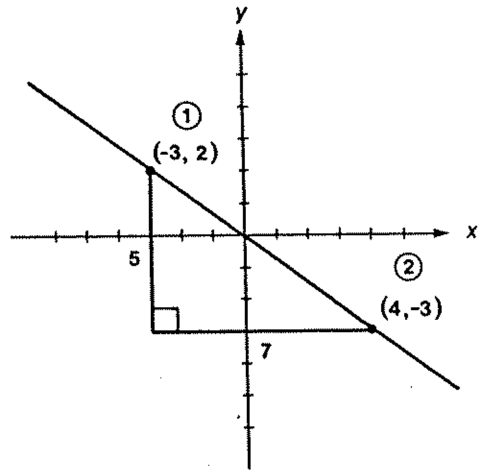
If the equation of a line is written in slope-intercept form, the constant term in the equation is the y intercept and the coefficient of x is the slope. The slope is the change in the y coordinate divided by the change in the x coordinate as we move from any point on the line to any other point on the line. The sign of the slope can be determined visually by remembering the following diagrams. **The little man always comes from the left side, as we show here.** He sees the first set of lines as uphill lines with positive slopes and the second set of lines as downhill lines with negative slopes.



Example RLD.A.1 Find the slope-intercept form of the equation of the line that passes through the points $(-3, 2)$ and $(4, -3)$.

Solution We graph the points and draw the line and a right triangle whose hypotenuse is the segment connecting the points.

[†] If the line is vertical it has no y intercept. A line that is not vertical and is not horizontal has a y intercept and also has an x intercept. The x intercept will be discussed in a later lesson.



$$\frac{|\text{Rise}|}{|\text{Run}|} = \frac{5}{7}$$

Slope is negative

$$\text{Slope} = -\frac{5}{7}$$

If we move from point 1 to point 2, the y coordinate changes from $+2$ to -3, a change of -5; and the x coordinate changes from -3 to $+4$, a change of $+7$. Thus, the slope is

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{-5}{+7} = -\frac{5}{7}$$

We get the same result if we move from point 2 to point 1. The y coordinate changes from -3 to $+2$, a change of $+5$; and the x coordinate changes from 4 to -3, a change of -7.

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{+5}{-7} = -\frac{5}{7}$$

The line appears to cross the y axis near the origin so the y intercept should be a number close to zero. We can find the exact value of the y intercept if we replace m in the linear equation $y = mx + b$ with $-\frac{5}{7}$ and then use the coordinates of either of the given points as replacements for x and y to solve for b.

$y = -\frac{5}{7}x + b$	equation
$2 = -\frac{5}{7}(-3) + b$	used $(-3, 2)$ for x and y
$\frac{14}{7} = \frac{15}{7} + b$	simplified
$-\frac{1}{7} = b$	solved

We find that b is $-\frac{1}{7}$, so we can write the equation of the line as

$$y = -\frac{5}{7}x - \frac{1}{7}$$

Example RLD.A.2 Find the equation of the line that passes through the point $(-2, 5)$ and is perpendicular to the line $3y + 4x = 2$.

Solution First we write the given line in slope-intercept form by solving for y.

$$y = -\frac{4}{3}x + \frac{2}{3}$$

A line that is perpendicular to this line would have a slope of $+\frac{3}{4}$ because the slopes of perpendicular lines are negative reciprocals. So we have

$$y = \frac{3}{4}x + b$$

Now we can use $(-2, 5)$ for x and y and solve for b.

$$5 = \frac{3}{4}(-2) + b \qquad \text{used } (-2, 5) \text{ for } x \text{ and } y$$

$$\frac{10}{2} = -\frac{3}{2} + b \qquad \text{simplified}$$

$$\frac{13}{2} = b \qquad \text{solved}$$

Now we have both the slope and the y intercept and can write the equation.

$$y = \frac{3}{4}x + \frac{13}{2}$$

RLD.B Complex fractions

A complex fraction is a fraction that contains more than one fraction line. Complex fractions are simplified by writing both the numerator and denominator as simple fractions, and then multiplying above and below by the reciprocal of the denominator.

Example RLD.B.1 Simplify: $\dfrac{\dfrac{a}{x^2} + \dfrac{b}{x}}{\dfrac{m}{x^2} + \dfrac{k}{xc}}$

Solution We begin by writing both the numerator and denominator as simple fractions.

$$\frac{\dfrac{a + bx}{x^2}}{\dfrac{mc + kx}{x^2c}}$$

Now we can multiply above and below by the reciprocal of the denominator.

$$\frac{\dfrac{a + bx}{x^2}}{\dfrac{mc + kx}{x^2c}} \cdot \frac{\dfrac{x^2c}{mc + kx}}{\dfrac{x^2c}{mc + kx}} = \frac{c(a + bx)}{mc + kx}$$

Example RLD.B.2 Simplify: $\dfrac{x}{a + \dfrac{m}{1 + \dfrac{c}{d}}}$

Solution We begin by writing $1 + \dfrac{c}{d}$ as a simple fraction.

$$\frac{x}{a + \dfrac{m}{\dfrac{d + c}{d}}}$$

Now we simplify the triple-decker fraction and get

$$\frac{x}{a+\dfrac{\frac{m}{1}}{\frac{d+c}{d}}\cdot\dfrac{\frac{d}{d+c}}{\frac{d}{d+c}}}=\frac{x}{a+\dfrac{md}{d+c}}$$

Now we add the two terms in the denominator and get

$$\frac{x}{\dfrac{a(d+c)+md}{d+c}}$$

and finish by multiplying above and below by the reciprocal of the denominator.

$$\frac{\frac{x}{1}}{\dfrac{a(d+c)+md}{d+c}}\cdot\frac{\dfrac{d+c}{a(d+c)+md}}{\dfrac{d+c}{a(d+c)+md}}=\frac{x(d+c)}{a(d+c)+md}$$

RLD.C Abstract equations

Equations composed of variables are sometimes called ***abstract equations.*** Often, it is necessary to rearrange one of these equations.

Example RLD.C.1 Solve $y=v\left(\frac{a}{x}+\frac{b}{mc}\right)$ for c.

Solution As the first step, we use the distributive property to clear the parentheses on the right.

$$y=\frac{va}{x}+\frac{vb}{mc}$$

Next we multiply every numerator by xmc, which is the least common multiple of the denominators.

$$xmc\cdot y=xmc\cdot\frac{va}{x}+xmc\cdot\frac{vb}{mc}$$

Now we can cancel the denominators, and we have left

$$xmcy=mcva+xvb$$

Since we are solving for c, we put all terms with c on one side of the equals sign (either side).

$$xmcy-mcva=xvb$$

Next we factor out c,

$$c(xmy-mva)=xvb$$

and finish by dividing both sides of the equation by the coefficient of c.

$$\frac{c(xmy-mva)}{xmy-mva}=\frac{xvb}{xmy-mva}\longrightarrow c=\frac{xvb}{xmy-mva}$$

RLD.D Checking division of polynomials

When polynomials are divided, the division can be checked by adding the quotient and the remainder fraction. We demonstrate this procedure in the following example.

Example RLD.D.1 Simplify $\frac{x^3 - 2}{x - 1}$ by dividing; then check the result by adding the quotient and the remainder fraction.

Solution First we perform the indicated division and remember to insert $0x^2$ and $0x$ to help with spacing.

$$\begin{array}{r} x^2 + x + 1 \\ x - 1 \overline{)\, x^3 + 0x^2 + 0x - 2} \\ \underline{x^3 - x^2} \qquad\qquad \\ x^2 + 0x \qquad \\ \underline{x^2 - x} \qquad \\ x - 2 \\ \underline{x - 1} \\ -1 \end{array}$$

Thus, we find that

$$\frac{x^3 - 2}{x - 1} = x^2 + x + 1 - \frac{1}{x - 1}$$

To check, we must write both parts of the answer with a denominator of $x - 1$ so the parts can be added. Thus we must multiply $x^2 + x + 1$ by $(x - 1)$ over $(x - 1)$.

$$(x^2 + x + 1) \cdot \frac{(x - 1)}{(x - 1)} - \frac{1}{x - 1}$$

We perform the multiplication on the left and get

$$\frac{x^3 + x^2 + x - x^2 - x - 1}{x - 1} - \frac{1}{x - 1}$$

and we simplify the numerators and get

$$\frac{x^3 - 2}{x - 1} \qquad \text{check}$$

Problem set D

1. The ratio of whats to hows was 7 to 3, and thrice the number of hows exceeded twice the number of whats by −40. How many were whats?
2. Fourteen percent of the blues were added to 20 percent of the greens for a total of 54. Also, 5 times the number of blues exceeded twice the number of greens by 100. How many were blue and how many were green?
3. A man stood on the top of a 500-ft building and measured a 20° angle of elevation to an airplane flying at an altitude of 2500 ft. What was the straight-line distance from the man to the airplane?
4. Find the equation of the line that passes through (−4, 5) and (−6, 3).
5. Find the equation of the line that passes through the point (−2, 5) and is perpendicular to the line $2x + 3y = 5$.

Simplify:

6. $\dfrac{\dfrac{x}{a^2}+\dfrac{b}{a}}{\dfrac{1}{a^2}-\dfrac{k}{ac}}$ 7. $\dfrac{p}{m+\dfrac{m}{1+\dfrac{b}{c}}}$ 8. $\dfrac{a}{x+\dfrac{y}{p+\dfrac{m}{c}}}$

9. Solve $y = m\left(\dfrac{a}{x}+\dfrac{b}{mc}\right)$ for c. 10. Solve $x = pm\left(\dfrac{1}{y}+\dfrac{a}{bd}\right)$ for d.

11. Divide $x^3 - 1$ by $x - 2$ and check the result by adding the quotient and the remainder fraction.

Simplify:

16. $\sqrt{-2}\sqrt{2}-\sqrt{2}i-\sqrt{-4}+i$ 17. $\dfrac{3+2\sqrt{3}}{4-12\sqrt{3}}$

18. $\dfrac{2i^2-3i+4i^3}{2i^5-3i-2i^2}$

Solve by completing the square:

19. $3x^2 = -6 - x$ 20. $4x + 7 = 2x^2$

21. Begin with $ax^2 + bx + c = 0$ and derive the quadratic formula.

22. Use the quadratic formula to solve $3x^2 = \dfrac{2}{5}x - 3$.

23. If $x, x + y \neq 0$ and if $x = \dfrac{1}{y+z}$, compare: *A.* $\dfrac{5}{x}$ *B.* $5(y+z)$

24. Given $-10 < z < -1$, compare: *A.* $\dfrac{1}{z^5}$ *B.* $\dfrac{1}{z^4}$

Solve:

25. $\begin{cases} x + 2y - z = -5 \\ 2x - y + z = 11 \\ x + y - z = -2 \end{cases}$ 26. $\begin{cases} 2x - z = 5 \\ 3x + 2y = 13 \\ y - 2z = 0 \end{cases}$

27. $\begin{cases} \dfrac{x}{2}-\dfrac{3y}{4}=\dfrac{5}{2} \\ \dfrac{x}{3}-\dfrac{2z}{5}=-\dfrac{8}{15} \\ 0.4y - 0.05z = -0.95 \end{cases}$

Simplify:

28. $b^2\sqrt[3]{ab^4}(ab^3)^{1/5}$ 29. $\dfrac{x^{a/3-2}y^{(b-2)/3}}{x^{2a}(y^{1/3})^{2a}}$

30. $(2x^{3/4}+y^{-1/2})(2x^{3/4}-y^{-1/2})$

REVIEW LESSON E *Review word problems · A(* *rs*

RLE.A Review word problems

Word problems are worked by transforming the written statements into mathematical equations and then solving the equations. **As many variables may be used as are convenient. For a unique solution, we must have as many independent equations as we have variables.**[†] The variables x, y, and z should be avoided because it is difficult to remember what they represent in a particular problem. Subscripted variables should be used because their meanings are easier to remember.

Word problems tend to be categorized into types according to the different thought processes required to find their solutions. Thus far, we have looked at simple problems whose solutions required the use of at most two variables. In this lesson, we will review the solution of problems that require the use of three variables in three equations. We will also review other types of problems. Some of these problems were selected because they require procedures that have a wide variety of applications. Other problems were selected because they represent types of problems that will be encountered in almost the same forms in chemistry and physics courses.

Example RLE.A.1 The number of blues was 4 less than the sum of the whites and the greens. Also the number of greens equaled the sum of the blues and the whites. How many of each were there if there were twice as many blues as whites?

Solution This problem can be worked by using three equations in three unknowns. We will use N_B, N_W, and N_G as the variables. The three equations are as follows:

(*a*) The number of blues was 4 less than the sum of the whites and the greens.

$$N_B + 4 = N_W + N_G$$

(*b*) The number of greens equaled the sum of the blues and the whites.

$$N_G = N_B + N_W$$

(*c*) There were twice as many blues as whites.

$$N_B = 2N_W$$

Note that in (*a*) we added 4 to the number of blues because there were 4 fewer blues. Also in (*c*) we multiplied the number of whites by 2 to equal the number of blues. When a statement tells how much greater or less a quantity is, addition or multiplication is required so that an equation (statement of equality) may be written. We begin by substituting $2N_W$ for N_B in equations (*a*) and (*b*).

$$\begin{array}{llcrl}
(a) & (2N_W) + 4 = N_W + N_G & \longrightarrow & N_W - N_G &= -4 \\
(b) & N_G = (2N_W) + N_W & \longrightarrow & -3N_W + N_G &= 0 \\
\hline
 & & & -2N_W &= -4 \\
 & & & N_W &= 2
\end{array}$$

Now N_B equals $2N_W$, so $N_B = 4$; and N_G equals $N_W + N_B$, so $N_G = 6$. Thus

$$\mathbf{N_W = 2 \qquad N_B = 4 \qquad N_G = 6}$$

[†] This is true for systems of linear equations if the domain for all variables and all coefficients is the set of real numbers.

Example RLE.A.2 The quarters, nickels, and dimes totaled 20, and their value was \$1.90. How many of each kind were there if there were 4 times as many nickels as quarters?

Solution There were 20 coins in all,

$$(a) \quad N_N + N_D + N_Q = 20$$

and their value was \$1.90.

$$(b) \quad 5N_N + 10N_D + 25N_Q = 190$$

There were 4 times as many nickels as quarters.

$$(c) \quad N_N = 4N_Q$$

We begin by using (*c*) to substitute for N_N in (*a*) and (*b*).

$$\begin{array}{llcrl} (a) & (4N_Q) + N_D + N_Q = 20 & \longrightarrow & N_D + \ \ 5N_Q = \ \ 20 & (d) \\ (b) & 5(4N_Q) + 10N_D + 25N_Q = 190 & \longrightarrow & 10N_D + 45N_Q = 190 & (e) \end{array}$$

Now we multiply (*d*) by -10 and add to (*e*).

$$\begin{array}{rcr} (-10)(d) & \longrightarrow & -10N_D - 50N_Q = -200 \\ (e) & \longrightarrow & 10N_D + 45N_Q = \ \ 190 \\ \hline & & -5N_Q = -10 \\ & & N_Q = \mathbf{2} \end{array}$$

$N_N = 4(2) = \mathbf{8}$, and since there were 20 in all, **10 were dimes.**

Example RLE.A.3 Reds varied directly as blues squared and inversely as greens. When there were 80 reds, there were 4 blues and 2 greens. How many reds were there when there were 8 blues and 10 greens?

Solution The problem can be worked as a variation problem. This approach is often used in physics books. The first sentence gives us the basic equation.

$$R = \frac{kB^2}{G}$$

where k is a constant. Next we must find k. We replace R with 80, B with 4, and G with 2, and solve for k.

$$80 = \frac{k(4)^2}{2} \quad \longrightarrow \quad k = 10$$

Now we replace k in the basic equation with 10.

$$R = \frac{10B^2}{G}$$

Since we have found k, we can complete the solution. To finish we replace B with 8 and G with 10 and solve for R.

$$R = \frac{10(8)^2}{10} \quad \longrightarrow \quad \mathbf{R = 64}$$

The second method is the ratio method. This approach is often used in chemistry books. The first sentence gives us the basic equation.

$$\frac{R_1}{R_2} = \frac{B_1^2 G_2}{B_2^2 G_1}$$

Now we make the required replacements and solve.

$$\frac{80}{R_2} = \frac{(4)^2(10)}{(8)^2(2)} \longrightarrow \frac{80}{R_2} = \frac{160}{128} \longrightarrow R_2 = 64$$

Example RLE.A.4 The sum of the digits of a two-digit counting number is 5. When the digits are reversed, the number is 9 greater than the original number. What was the original number?

Solution The counting numbers are the positive integers. The sum of the digits is 5. If we use U for the units digit and T for the tens digit, we get

$$(a) \quad U + T = 5$$

The value of the units digit is U and of the tens digit is $10T$, but when the digits are reversed the values will be $10U$ and T.

	ORIGINAL NUMBER		NEW NUMBER MINUS 9
(b)	$10T + U$	$=$	$T + 10U - 9$

which simplifies to

$$9T - 9U = -9 \longrightarrow T - U = -1$$

Now we substitute from equation (*a*) and solve.

$$\begin{aligned} (5 - U) - U &= -1 && \text{substituted for } T \\ 5 - 2U &= -1 && \text{added} \\ -2U &= -6 && \text{added } -5 \\ U &= 3 \end{aligned}$$

Since $U + T = 5$, $T = 2$ and the original number was **23**.

Example RLE.A.5 To get 1000 gallons (gal) of a mixture that was 56% alcohol, it was necessary to mix some 20% alcohol solution with some 80% alcohol solution. How much of each was required?

Solution We decide to make the statement about alcohol.

$$\text{Alcohol}_1 + \text{alcohol}_2 = \text{alcohol total}$$

Next we use parentheses as mixture containers.

$$(\quad) + (\quad) = (\quad)$$

We pour in some of the first mixture (P_N), and dump in some of the second mixture (D_N) for a total of 1000.

$$(P_N) + \quad (D_N) = \quad (1000)$$

Now we multiply by the proper decimals so that each entry represents alcohol.

$$(a) \quad 0.2(P_N) + 0.8(D_N) = 0.56(1000)$$

This equation has two unknowns so we need another equation, which is

$$(b) \quad P_N + D_N = 1000$$

Now we substitute to solve.

$$\begin{aligned} 0.2(1000 - D_N) + 0.8D_N &= 0.56(1000) && \text{substituted} \\ 200 - 0.2D_N + 0.8D_N &= 560 && \text{multiplied} \end{aligned}$$

Now we eliminate the decimals by multiplying by 10.

$$2000 - 2D_N + 8D_N = 5600$$
$$6D_N = 3600$$
$$D_N = 600 \text{ gal of } 80\% \text{ alcohol}$$

Since the total was 1000 gal, we need 400 gal of 20% alcohol.

Example RLE.A.6 How many liters of a 64% glycol solution must be added to 77 liters of a 23% glycol solution to get a 42% glycol solution?

Solution The solution to this problem is not difficult if a calculator is used to help with the arithmetic. We will make the statement about glycol and then insert the indicated quantities in the parentheses used as mixture containers.

$$\text{Glycol}_1 + \text{glycol added} = \text{glycol final}$$
$$(77) \quad + \quad (P_N) \quad = (77 + P_N)$$

The mixture entries indicate the amount of mixture. It is important to use symbols such as P_N or D_N for the amount of solution added. **Avoid using G for glycol added because the mixture added was not all glycol.** Next we multiply each of the mixture container entries by the proper decimal number so that the product will equal the amount of glycol for each step.

$$0.23(77) + 0.64(P_N) = 0.42(77 + P_N)$$

We use a calculator to permit a quick solution to this equation.

$$17.71 + 0.64P_N = 32.34 + 0.42P_N \qquad \text{multiplied}$$
$$0.22P_N = 14.63 \qquad \text{rearranged}$$
$$P_N = 66.5 \text{ liters} \qquad \text{divided}$$

Example RLE.A.7 The weight of the carbon (C) in the container of C_3H_7Cl was 113 grams. What was the total weight of the compound? (C, 12; H, 1; Cl, 35)

Solution This is a ratio problem. The gram atomic weights are given above in parentheses.

Carbon	12 × 3 = 36
Hydrogen	1 × 7 = 7
Chlorine	35 × 1 = 35
Total	78

Thus, the ratio of the carbon to the total weight is 36 to 78, and the carbon weighed 113 grams.

$$\frac{C}{T} = \frac{36}{78} \longrightarrow \frac{113}{T} = \frac{36}{78} \longrightarrow T = 244.8 \text{ grams}$$

Problem set E

1. The number of blues was 7 less than the sum of the whites and the greens. The number of greens was 1 greater than the sum of the blues and the whites. How many of each kind were there if there were 3 times as many greens as blues?
2. The quarters, nickels, and dimes totaled 20, and their value was \$2.05. How many of each kind were there if there were 3 times as many dimes as quarters?
3. Reds varied directly as blues and inversely as greens squared. When there were 5 reds, there were 2 blues and 4 greens. How many reds were there when there were 4 blues and 4 greens?
4. The sum of the digits of a two-digit counting number is 13. When the digits are reversed, the new number is 45 greater than the original number. What was the original number?
5. How much of a 14% iodine solution should be added to 89 ounces of a 47% iodine solution to get a 29% solution?
6. The weight of the chlorine (Cl) in the container of C_3H_7Cl was 400 grams. What was the total weight of the compound? (C, 12; H, 1; Cl, 35)
7. Find the equation of the line that passes through the point $(-2, 3)$ and is perpendicular to the line $5x - 2y + 4 = 0$.

Simplify:

8. $\dfrac{\dfrac{m^2}{a^2} + \dfrac{7y}{x}}{\dfrac{p^2}{ax} - \dfrac{3}{a^2}}$

9. $\dfrac{m}{a + \dfrac{b}{1 + \dfrac{c}{d}}}$

10. $\dfrac{k}{a + \dfrac{b}{x + \dfrac{c}{d}}}$

11. Solve for c: $x = kb\left(\dfrac{1}{c} - \dfrac{a}{x}\right)$

12. Solve for k: $mc = p\left(\dfrac{a}{cm} + \dfrac{2}{kc}\right)$

13. Divide $x^4 - 2$ by $x^2 - 1$ and check.

Simplify:

16. $\sqrt{2}\sqrt{-2} + \sqrt{-3}\sqrt{-3} + \sqrt{-4} - i$

17. $\dfrac{4 + 2\sqrt{12}}{6 - \sqrt{48}}$

18. $\dfrac{3i^3 - 2i + i^2}{1 - 4i^3}$

Solve by completing the square:

19. $4x^2 = -5 - 2x$

20. $3x + 7 = 4x^2$

21. Derive the quadratic formula.

22. Use the quadratic formula to solve $3x^2 = -4 + 2x$.

23. Compare: *A.* 5 *B.* $\sqrt{9} + \sqrt{16}$

24. Compare: *A.* 5 *B.* $\sqrt{3^2 + 2^2}$

Solve:

25. $\begin{cases} 2x + 2y - z = -1 \\ x + y - 3z = -8 \\ 2x - y + z = 8 \end{cases}$

26. $\begin{cases} 2x - z = 10 \\ y + 2z = -2 \\ 3x - 2y = 8 \end{cases}$

27. $\begin{cases} \dfrac{3}{2}x + \dfrac{4}{3}y = \dfrac{1}{2} \\ -\dfrac{x}{2} + \dfrac{z}{4} = -1 \\ 0.2y - 0.04z = -0.68 \end{cases}$

Simplify:

28. $x^3\sqrt{x^3y}(ab^{1/3})^2$

29. $\dfrac{x^{a/2-3}y^{(b-3)/2}}{x^{3a}(y^{1/3})^{2a}}$

30. $(2x^{a/2} + y^{b/2})(2x^{a/2} - y^{b/2})$

REVIEW LESSON F *Nonlinear systems · Factoring exponentials · Sum and difference of two cubes*

RLF.A Nonlinear systems

A nonlinear system of equations contains one or more nonlinear equations. Nonlinear systems are solved by using substitution and elimination just as we do when we solve systems of linear equations. Solving nonlinear systems often involves the solution of a quadratic equation as the final step. These quadratic equations are seldom factorable, and we usually solve them by using the quadratic formula without even trying the factor method of solution.

Example RLF.A.1 Solve: $\begin{cases} (a) & x^2 + y^2 = 9 & \text{(circle)} \\ (b) & y - x = 1 & \text{(line)} \end{cases}$

Solution We will begin by solving equation (*b*) for *y* and then we will square both sides.

$$y - x = 1 \qquad \text{equation } (b)$$
$$y = x + 1 \qquad \text{solved for } y$$
$$y^2 = x^2 + 2x + 1 \qquad \text{squared both sides}$$

Now we will replace y^2 in equation (*a*) with $x^2 + 2x + 1$.

$$x^2 + (x^2 + 2x + 1) = 9 \qquad \text{substituted}$$

$$2x^2 + 2x - 8 = 0 \qquad \text{simplified}$$

$$x^2 + x - 4 = 0 \qquad \text{divided by 2}$$

This equation cannot be solved by factoring, so we will use the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \longrightarrow x = \frac{-1 \pm \sqrt{1 - 4(1)(-4)}}{2} \longrightarrow x = -\frac{1}{2} \pm \frac{\sqrt{17}}{2}$$

$$\text{which means} \qquad x = -\frac{1}{2} + \frac{\sqrt{17}}{2} \qquad \text{and} \qquad x = -\frac{1}{2} - \frac{\sqrt{17}}{2}$$

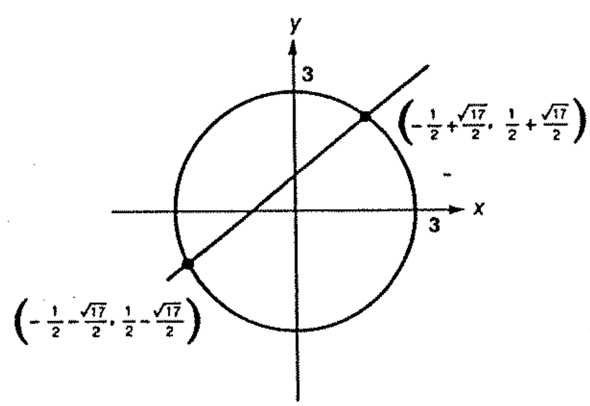
Now we could use either equation (*a*) or equation (*b*) to find the values of *y*. We will use the equation of the line (*b*) to find *y* because this equation has no squared terms and is easier to use, and also because the use of the quadratic equation might lead to spurious solutions.

$y = x + 1$	$y = x + 1$	equation (*b*)
$y = \left(-\frac{1}{2} + \frac{\sqrt{17}}{2}\right) + 1$	$y = \left(-\frac{1}{2} - \frac{\sqrt{17}}{2}\right) + 1$	substituted
$y = \frac{1}{2} + \frac{\sqrt{17}}{2}$	$y = \frac{1}{2} - \frac{\sqrt{17}}{2}$	simplified

Thus, the ordered pairs of *x* and *y* that satisfy the given system are:

$$\left(-\frac{1}{2} + \frac{\sqrt{17}}{2}, \frac{1}{2} + \frac{\sqrt{17}}{2}\right) \qquad \text{and} \qquad \left(-\frac{1}{2} - \frac{\sqrt{17}}{2}, \frac{1}{2} - \frac{\sqrt{17}}{2}\right)$$

The graph of the line and the circle are shown here. We will study the graphs of circles and other conics in later lessons. Here we are concentrating on the algebra of the solutions.



Example RLF.A.2 Solve the system: $\begin{cases} (a) & x^2 + y^2 = 9 & \text{(circle)} \\ (b) & 2x^2 - y^2 = -6 & \text{(hyperbola)} \end{cases}$

Solution This system can be solved by using either substitution or elimination. We must be careful to get all the answers because this circle and hyperbola intersect at four different points. We decide to use elimination. We can eliminate the y^2 terms if we add the equations just as they are. If we do this, we get

$$3x^2 = 3 \qquad \text{added}$$
$$x^2 = 1 \qquad \text{divided by 3}$$

Here we must be careful because this equation has both $+1$ and -1 as solutions.

$$x = \pm\sqrt{1} \longrightarrow x = 1, -1$$

Now we must use these values of x one at a time to solve for y. We will use equation (a) and begin by letting x equal $+1$.

$$(1)^2 + y^2 = 9 \qquad \text{substituted (1) for } x$$
$$y^2 = 8 \qquad \text{added } -1$$
$$y = \pm 2\sqrt{2} \qquad \text{solved}$$

Thus, there are two points of intersection when $x = 1$. So two solutions of our system are

$$(\mathbf{1}, \mathbf{2\sqrt{2}}) \quad \text{and} \quad (\mathbf{1}, \mathbf{-2\sqrt{2}})$$

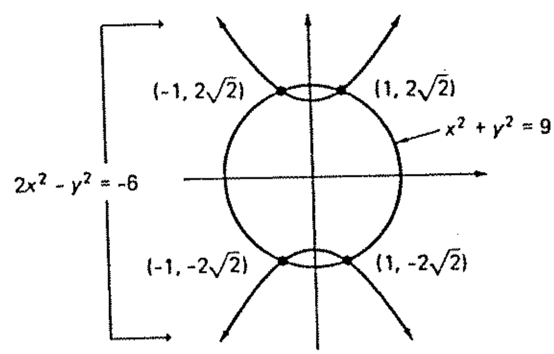
Next, we find the values of y that pair with a value of -1 for x. Again we use equation (a).

$$(-1)^2 + y^2 = 9 \qquad \text{substituted } (-1) \text{ for } x$$
$$y^2 = 8 \qquad \text{added } -1$$
$$y = \pm 2\sqrt{2} \qquad \text{solved}$$

Thus, our other two solutions to the system are

$$(\mathbf{-1}, \mathbf{2\sqrt{2}}) \quad \text{and} \quad (\mathbf{-1}, \mathbf{-2\sqrt{2}})$$

Here we show the graphs of the two curves and note that there are four points where the curves intersect. Do not worry about the graphs now. They will be discussed in later lessons.



Example RLF.A.3 Solve the system: $\begin{cases} xy = -4 \\ y = -x - 2 \end{cases}$

Solution We decide to use substitution. Thus, in the top equation, we substitute $-x - 2$ for y:

$$x(-x - 2) = -4 \qquad \text{substituted}$$
$$-x^2 - 2x = -4 \qquad \text{multiplied}$$
$$x^2 + 2x - 4 = 0 \qquad \text{rearranged}$$

We will use the quadratic formula to solve this equation.

$$x = \frac{-2 \pm \sqrt{4 - 4(1)(-4)}}{2} = \frac{-2 \pm \sqrt{20}}{2} = -1 \pm \sqrt{5}$$

We find that the values of x that satisfy the equation are $-1+\sqrt{5}$ and $-1-\sqrt{5}$. We will use both these values of x in the linear equation and solve for the paired values of y.

IF $x = -1+\sqrt{5}$	IF $x = -1-\sqrt{5}$
$y = -(-1+\sqrt{5})-2$	$y = -(-1-\sqrt{5})-2$
$y = 1-\sqrt{5}-2$	$y = 1+\sqrt{5}-2$
$y = -1-\sqrt{5}$	$y = -1+\sqrt{5}$

Thus, the ordered pairs of x and y that satisfy this system of nonlinear equations are

$$(-1+\sqrt{5}, -1-\sqrt{5}) \quad \text{and} \quad (-1-\sqrt{5}, -1+\sqrt{5})$$

RLF.B Factoring exponentials

Expressions in which a base is raised to a power are often called ***exponential expressions*** or just ***exponentials.*** The word *exponential* also has a more restrictive definition that is explained in Lesson 2. When two quantities are multiplied, each of the quantities is called a ***factor.*** The reverse process is called ***factoring*** because when we factor, we break up an algebraic sum into an indicated product of two or more factors.

Example RLF.B.1 Factor: $3x^{2n+2}+12x^{3n+3}$

Solution Sometimes it helps if we rewrite exponential expressions whose exponents are complicated. If we do, we get

$$3\cdot x^n\cdot x^n\cdot x^2+3\cdot 4\cdot x^n\cdot x^n\cdot x^n\cdot x^3$$

We see that the common factor is $3x^nx^nx^2$. Thus, we can write the expression in factored form as

$$3x^{2n}x^2(1+4x^nx) = \mathbf{3x^{2n+2}(1+4x^{n+1})}$$

Example RLF.B.2 Simplify: $\dfrac{x^{2a}-y^{2b}}{x^a+y^b}$

Solution We recognize that the numerator is really the difference of two squares and can be factored.

$$\frac{(x^a)^2-(y^b)^2}{x^a+y^b} = \frac{(x^a+y^b)(x^a-y^b)}{x^a+y^b} = \mathbf{x^a-y^b}$$

RLF.C Sum and difference of two cubes

The sum of two cubes can be factored and the difference of two cubes can be factored.

$$a^3+b^3=(a+b)(a^2-ab+b^2) \qquad a^3-b^3=(a-b)(a^2+ab+b^2)$$

Using the forms for factoring the sum or difference of two cubes is required, and disuse encourages one to forget the factored forms. Yet, if one can remember that a^3+b^3 is divisible by $a+b$ and that a^3-b^3 is divisible by $a-b$, it is possible to do the long divisions to find the other factors, as we show here.

$$\begin{array}{r|l} & a^2-ab+b^2 \\ \hline a+b & a^3 \qquad\qquad + b^3 \\ & \underline{a^3+a^2b} \\ & -a^2b \\ & \underline{-a^2b-ab^2} \\ & ab^2+b^3 \\ & \underline{ab^2+b^3} \end{array} \qquad \begin{array}{r|l} & a^2+ab+b^2 \\ \hline a-b & a^3 \qquad\qquad - b^3 \\ & \underline{a^3-a^2b} \\ & a^2b \\ & \underline{a^2b-ab^2} \\ & ab^2-b^3 \\ & \underline{ab^2-b^3} \end{array}$$

Thus, we see that we can factor as follows:

$$(1)\quad a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$
$$(2)\quad a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

To extend these forms to more complicated expressions, some people find that it is helpful to think F for *first thing* and S for *second thing* instead of a and b. If we do this in the above, we get

$$(1')\quad F^3 + S^3 = (F + S)(F^2 - FS + S^2)$$
$$(2')\quad F^3 - S^3 = (F - S)(F^2 + FS + S^2)$$

Example RLF.C.1 Factor $x^3y^3 - p^3$.

Solution We recognize that this expression can be written as the difference of two cubes:

$$(xy)^3 - (p)^3$$

The first thing that is cubed is xy and the second thing that is cubed is p. If we use form (2′) above,

$$(2')\quad F^3 - S^3 = (F - S)(F^2 + FS + S^2)$$

and replace F with xy and S with p, we can write the given expression in factored form.

$$x^3y^3 - p^3 = (xy - p)(x^2y^2 + xyp + p^2)$$

Example RLF.C.2 Factor $8m^3y^6 + x^3$.

Solution We recognize this as the sum of two cubes.

$$(2my^2)^3 + (x)^3$$

and note that the first thing that is cubed is $2my^2$ and that the second thing that is cubed is x. Thus, if we use form (1′),

$$(1')\quad F^3 + S^3 = (F + S)(F^2 - FS + S^2)$$

and replace F with $2my^2$ and S with x, we get

$$8m^3y^6 + x^3 = (2my^2 + x)(4m^2y^4 - 2my^2x + x^2)$$

Problem set F

1. The number of reds was 11 fewer than the sum of the blues and whites. The number of whites was 3 fewer than the sum of the reds and blues. How many of each were there if the number of whites was 1 greater than the number of blues?
2. There were 30 nickels, dimes, and pennies in all, and their value was $1.35. If there were twice as many pennies as dimes, how many coins of each kind were there?
3. The number of boys varied inversely as the number of teachers and directly as the number of girls squared. When there were 200 boys, there were 10 girls and 20 teachers. How many boys were there when there were 2 teachers and 8 girls?
4. The sum of the digits of a two-digit counting number is 5. When the digits are reversed, the new number is 27 greater than the original number. What are the two numbers?
5. How many liters of a 90% alcohol solution should be mixed with how many liters of a 58% alcohol solution to make 20 liters of a 78% alcohol solution?
6. How many milliliters of a $13\frac{1}{2}$% iodine solution should be mixed with 370 ml of a 6% iodine solution to get a 10% iodine solution?

7. Find the equation of the line that passes through $(-2, 4)$ that is parallel to the line $3x + 2y - 4 = 0$.

Solve:

8. $\begin{cases} x^2 + y^2 = 9 \\ y - x = 1 \end{cases}$ 9. $\begin{cases} x^2 + y^2 = 9 \\ 2x^2 - y^2 = -6 \end{cases}$ 10. $\begin{cases} xy = -4 \\ y = -x - 2 \end{cases}$

Factor:

11. $8x^6y^3 + p^3$ 12. $27x^{12}y^6 - z^9$

Simplify:

13. $\dfrac{\dfrac{n^2}{b^3} + \dfrac{7p}{a}}{\dfrac{r^2}{ba} - \dfrac{4}{b^3}}$ 14. $\dfrac{m}{c + \dfrac{f}{2 + \dfrac{g}{h}}}$

15. Solve for x: $k^2 = \dfrac{1}{bc}\left(\dfrac{x}{3} - \dfrac{6y^3}{d}\right)$

16. Divide $x^3 - 3x^2 - 3$ by $x - 2$ and check.

Simplify:

18. $\sqrt{3}\sqrt{-3} - \sqrt{-2}\sqrt{-2} + \sqrt{-4} + 5i + \sqrt{-3}$

19. $\dfrac{3 + 2\sqrt{12}}{2 - 4\sqrt{48}}$ 20. $\dfrac{3i^4 - 2i^2 + i^3}{1 - 3i^2 + \sqrt{-9}}$

Solve by completing the square:

21. $3x^2 = -4 + 3x$ 22. $-6x - 9 = 2x^2$

23. If $0 < x < 20$ and $0 < y < 24$, compare: A. $\dfrac{x}{y} + 5$ B. $\dfrac{y}{x} + 10$

24. If $x \in R$ and $y \in R$, and $y \neq 0$, compare: A. $\dfrac{x}{y}$ B. $1 + \dfrac{x}{y}$

Solve:

25. $\begin{cases} 2x + 3y - z = 6 \\ 3x - y + z = 1 \\ x + y + z = 1 \end{cases}$ 26. $\begin{cases} 2x - z = 1 \\ y + 2z = 7 \\ 3x - y = 5 \end{cases}$

27. $\begin{cases} \dfrac{3}{4}x + \dfrac{5}{2}y = 8 \\ -\dfrac{x}{2} - \dfrac{z}{2} = -\dfrac{5}{2} \\ 0.25y - 0.4z = 0.1 \end{cases}$

Simplify:

28. $x^3\sqrt{x^4y^3}\,(x^3y)^{1/3}$ 29. $\dfrac{x^{a/3-2}y^{(b-2)/3}}{x^{2a}(y^{2/3})^a}$

30. $(3x^{a/2} + 2y^{b/3})(2x^{a/2} - 2y^{b/3})$